Classical Examples

Let $H(\mathbb{C}) = \{f : \mathbb{C} \to \mathbb{C} | f \text{ analytic}\}$ = the set of all entire functions $f : \mathbb{C} \to \mathbb{C}$.

Classical Examples

Let $H(\mathbb{C}) = \{f : \mathbb{C} \to \mathbb{C} | f \text{ analytic}\}$ = the set of all entire functions $f : \mathbb{C} \to \mathbb{C}$.

Convergence: \$f_n \to f$ in $H(\mathbb{C})$" means \$f_n \to f$ uniformly on compact subsets of $\mathbb{C}$."

Classical Examples

Let $H(\mathbb{C}) = \{f : \mathbb{C} \to \mathbb{C} | f \text{ analytic}\}$ = the set of all entire functions $f : \mathbb{C} \to \mathbb{C}$.

Convergence: \$f_n \to f$ in $H(\mathbb{C})$" means \$f_n \to f$ uniformly on compact subsets of $\mathbb{C}$."

- Birkho (1929): There is a function $f \in H(\mathbb{C})$ so that $\{f(z), f(z+1), f(z+2), f(z+3) \dots\}$ is dense in $H(\mathbb{C})$.

Classical Examples

Let $H(\mathbb{C}) = \{f : \mathbb{C} \to \mathbb{C} | f \text{ analytic}\}$ = the set of all entire functions $f : \mathbb{C} \to \mathbb{C}$.

Convergence: \$f_n \to f$ in $H(\mathbb{C})$" means \$f_n \to f$ uniformly on compact subsets of $\mathbb{C}$."

- Birkho (1929): There is a function $f \in H(\mathbb{C})$ so that $\{f(z), f(z+1), f(z+2), f(z+3) \dots\}$ is dense in $H(\mathbb{C})$.

- G. R. MacLane (1952): There is a function $f \in H(\mathbb{C})$ so that $\{f(z), f'(z), f''(z), f'''(z) \dots\}$ is dense in $H(\mathbb{C})$.

Classical Examples

Let $H(\mathbb{C}) = \{f : \mathbb{C} \to \mathbb{C} | f \text{ analytic}\}$ = the set of all entire functions $f : \mathbb{C} \to \mathbb{C}$.

Convergence: \$f_n \to f$ in $H(\mathbb{C})$" means \$f_n \to f$ uniformly on compact subsets of $\mathbb{C}$."

- Birkho (1929): There is a function $f \in H(\mathbb{C})$ so that $\{f(z), f(z+1), f(z+2), f(z+3) \ldots\}$ is dense in $H(\mathbb{C})$.

- G. R. MacLane (1952): There is a function $f \in H(\mathbb{C})$ so that $\{f(z), f'(z), f''(z), f'''(z) \ldots\}$ is dense in $H(\mathbb{C})$.

Let $p \geq 1$, and $B : \ell^p \to \ell^p$ be the unilateral backward shift de ned by $B(a_0, a_1, a_2, \ldots) = (a_1, a_2, a_3, \ldots)$.

- Rolewicz (1969): If $t \in (1, \infty)$, then there exists a vector x in ℓ^p so that $\{x, (tB)x, (tB)^2x, (tB)^3x, \ldots\}$ is dense in ℓ^p.

Hypercyclicity Criterion

Let X be a separable, infinite-dimensional Banach space over $\mathbb{C}$, and $B(X) = \{T : X \to X | T \text{ is bounded and linear}\}$.

Hypercyclicity Criterion

Let X be a separable, infinite-dimensional Banach space over $\mathbb{C}$, and $B(X) = \{T : X \to X | T$ is bounded and linear$\}$.

Definition. A bounded linear operator T in $B(X)$ is <u>*hypercyclic*</u> if there is a vector x whose orbit $\mathrm{orb}(T, x) = \{x, Tx, T^2x, T^3x, \ldots\}$ is dense in X. Such a vector x is called a <u>*hypercyclic vector*</u>.

Hypercyclicity Criterion

Let X be a separable, infinite-dimensional Banach space over $\mathbb{C}$, and $B(X) = \{T : X \to X | T \text{ is bounded and linear}\}$.

Definition. A bounded linear operator T in $B(X)$ is <u>*hypercyclic*</u> if there is a vector x whose orbit $\text{orb}(T, x) = \{x, Tx, T^2x, T^3x, \ldots\}$ is dense in X. Such a vector x is called a <u>*hypercyclic vector*</u>.

- Kitai (1982), Gethner and Shapiro (1987): $T : X \to X$ is hypercyclic if there is a dense set D of X and T has a right inverse S so that $T^n x \to 0$ and $S^n x \to 0$ for each vector $x \in D$.

The Invariant Subspace Problem

$\overline{\text{orb}(T,x)} = \overline{\{x, Tx, T^2x, T^3x, \ldots\}}$ = the smallest closed invariant subset containing x.

The Invariant Subspace Problem

$\overline{\text{orb}(T, x)} = \overline{\{x, Tx, T^2x, T^3x, \ldots\}}$ = the smallest closed invariant subset containing x.

$\overline{\text{span orb}(T, x)} = \overline{\{p(T)x : p \text{ is a polynomial}\}}$ = the smallest closed invariant subspace containing x.

$\overline{\text{orb}(T,x)} = \overline{\{x, Tx, T^2x, T^3x, \dots\}}$ = the smallest closed invariant subset containing x.

$\overline{\text{span orb}(T,x)} = \overline{\{p(T)x : p \text{ is a polynomial}\}}$ = the smallest closed invariant subspace containing x.

Invariant Subspace Problem (1920s, still open today): Must every bounded linear operator $T : H \to H$ on an infinite dimensional separable Hilbert space H have a nontrivial invariant subspace?

The Invariant Subspace Problem

$\overline{\mathrm{orb}(T, x)} = \overline{\{x, Tx, T^2x, T^3x, \ldots\}}$ = the smallest closed invariant subset containing x.

$\overline{\mathrm{span}\ \mathrm{orb}(T, x)} = \overline{\{p(T)x : p \text{ is a polynomial}\}}$ = the smallest closed invariant subspace containing x.

Invariant Subspace Problem (1920s, still open today): Must every bounded linear operator $T : H \to H$ on an in nite dimensional separable Hilbert space H have a nontrivial invariant subspace?

- En o (1987): Not if the space is ℓ^1. That is, there is an operator T on ℓ^1 for which every nonzero vector x has the property that $\overline{\mathrm{span}\ \mathrm{orb}(T, x)} = \ell^1$.

The Invariant Subspace Problem

$\overline{\text{orb}(T,x)} = \overline{\{x, Tx, T^2x, T^3x, \dots\}}$ = the smallest closed invariant subset containing x.

$\overline{\text{span orb}(T,x)} = \overline{\{p(T)x : p \text{ is a polynomial}\}}$ = the smallest closed invariant subspace containing x.

Invariant Subspace Problem (1920s, still open today): Must every bounded linear operator $T : H \to H$ on an in nite dimensional separable Hilbert space H have a nontrivial invariant subspace?

- En o (1987): Not if the space is ℓ^1. That is, there is an operator T on ℓ^1 for which every nonzero vector x has the property that $\overline{\text{span orb}(T,x)} = \ell^1$.
- Read (1989): There is an operator T on ℓ^1 with no nontrivial closed invariant subset. That is, every nonzero vector x has the property that $\overline{\text{orb}(T,x)} = \ell^1$.

Nonhypercyclic Operators

If X is @0 [(0059.9e10004(dimensional,1000 [no1000 [orato827(r)1(s8]TJ

Nonhypercyclic Operators

If X is nite dimensional, no operator T on X is hypercyclic.

Proof: Take $X = \mathbb{C}^n$. The adjoint $T^* : \mathbb{C}^n \to \mathbb{C}^n$ has an eigenvalue $\alpha \in \mathbb{C}$. Suppose $T^* y = \alpha y$.

Nonhypercyclic Operators

If X is finite dimensional, no operator T on X is hypercyclic.

Proof: Take $X = \mathbb{C}^n$. The adjoint $T^*: \mathbb{C}^n \to \mathbb{C}^n$ has an eigenvalue $\alpha \in \mathbb{C}$. Suppose $T^*y = \alpha y$. Then for any vector $x \in \mathbb{C}^n$,

$$< T^n x, y > = < x, T^{*n} y > = < x, \alpha^n y > = \alpha^n < x, y >,$$

which cannot be dense in $\mathbb{C}$. $\square$

Nonhypercyclic Operators

If X is nite dimensional, no operator T on X is hypercyclic.

Proof: Take $X = \mathbb{C}^n$. The adjoint $T^* : \mathbb{C}^n \to \mathbb{C}^n$ has an eigenvalue $\alpha \in \mathbb{C}$. Suppose $T^* y = \alpha y$. Then for any vector $x \in \mathbb{C}^n$,

$$< T^n x, y > = < x, T^{*n} y > = < x, \alpha^n y > = \alpha^n < x, y >,$$

which cannot be dense in $\mathbb{C}$. $\square$

Back to the case when X is in nite dimensional.

Nonhypercyclic Operators

If X is nite dimensional, no operator T on X is hypercyclic.

Proof: Take $X = \mathbb{C}^n$. The adjoint $T^* : \mathbb{C}^n \to \mathbb{C}^n$ has an eigenvalue $\alpha \in \mathbb{C}$. Suppose $T^* y = \alpha y$. Then for any vector $x \in \mathbb{C}^n$,

$$< T^n x, y > = < x, T^{*n} y > = < x, \alpha^n y > = \alpha^n < x, y >,$$

which cannot be dense in $\mathbb{C}$. $\square$

Back to the case when X is in nite dimensional.

If $F : X \to X$ has nite rank (that is, $\dim \operatorname{ran} F < \infty$), then $I + F$ is not hypercyclic.

Nonhypercyclic Operators

If X is nite dimensional, no operator T on X is hypercyclic.

Proof: Take $X = \mathbb{C}^n$. The adjoint $T^* : \mathbb{C}^n \to \mathbb{C}^n$ has an eigenvalue $\alpha \in \mathbb{C}$. Suppose $T^* y = \alpha y$. Then for any vector $x \in \mathbb{C}^n$,

$$< T^n x, y > = < x, T^{*n} y > = < x, \alpha^n y > = \alpha^n < x, y >,$$

which cannot be dense in $\mathbb{C}$. $\square$

Back to the case when X is in nite dimensional.

If $F : X \to X$ has nite rank (that is, $\dim \operatorname{ran} F < \infty$), then $I + F$ is not hypercyclic.

If $T : X \to X$ is compact, then T is not hypercyclic.

Nonhypercyclic Operators

If X is nite dimensional, no operator T on X is hypercyclic.

Proof: Take $X = \mathbb{C}^n$. The adjoint $T^* : \mathbb{C}^n \to \mathbb{C}^n$ has an eigenvalue $\alpha \in \mathbb{C}$. Suppose $T^* y = \alpha y$. Then for any vector $x \in \mathbb{C}^n$,

$$< T^n x, y > = < x, T^{*n} y > = < x, \alpha^n y > = \alpha^n < x, y >,$$

which cannot be dense in $\mathbb{C}$. $\square$

Back to the case when X is in nite dimensional.

If $F : X \to X$ has nite rank (that is, $\dim \operatorname{ran} F < \infty$), then $I + F$ is not hypercyclic.

If $T : X \to X$ is compact, then T is not hypercyclic.

If X is a Hilbert space, no normal operator is hypercyclic.

Hypercyclic vectors

Suppose $\{x_j : j \geq 1\}$ is a countable dense subset of X, and x is a vector in X. For x to be a hypercyclic vector, the following must hold:

For all x_j and for all $\epsilon > 0$, there is a integer n such that $\|T^n x - x_j\| < \epsilon$;

Hypercyclic vectors

Suppose $\{x_j : j \geq 1\}$ is a countable dense subset of X, and x is a vector in X. For x to be a hypercyclic vector, the following must hold:

For all x_j and for all $\epsilon > 0$, there is a integer n such that $\|T^n x - x_j\| < \epsilon$;

Equivalently, $x \in T^{-n}B(x_j, \epsilon)$.

Hypercyclic vectors

Suppose $\{x_j : j \geq 1\}$ is a countable dense subset of X, and x is a vector in X. For x to be a hypercyclic vector, the following must hold:

For all x_j and for all $\epsilon > 0$, there is a integer n such that $\|T^n x - x_j\| < \epsilon$;

Equivalently, $x \in T^{-n}B(x_j, \epsilon)$.

Let $\mathcal{HC}(T) = \{x \in X | x$ is a hypercyclic vector for $T\}$.

Hypercyclic vectors

Suppose $\{x_j : j \geq 1\}$ is a countable dense subset of X, and x is a vector in X. For x to be a hypercyclic vector, the following must hold:

For all x_j and for all $\epsilon > 0$, there is a integer n such that $\|T^n x - x_j\| < \epsilon$;

Equivalently, $x \in T^{-n}B(x_j, \epsilon)$.

Let $\mathcal{HC}(T)$ = *1*

A Basic Zero-One Law for Hypercyclic Vectors

- Kitai (1982): For any operator T in $B(X)$, either $\mathcal{HC}(T)$ is either ϕ or a dense G set.

A Basic Zero-One Law for Hypercyclic Vectors

- Kitai (1982): For any operator T in $B(X)$, either $\mathcal{HC}(T)$ is either ϕ or a dense G set.

Baire Category Theorem $\Longrightarrow$

If $\{T_n : X \to X | n \geq 1\}$ is a countable collection of hypercyclic operators, then their set of <u>*common hypercyclic vectors*</u>

$$\bigcap_{n=1}^{\infty} \mathcal{HC}(T_n)$$

is a dense G set.

A Basic Zero-One Law for Hypercyclic Vectors

- Kitai (1982): For any operator T in $B(X)$, either $\mathcal{HC}(T)$ is either ϕ or a dense G set.

Baire Category Theorem $\Longrightarrow$

If $\{T_n : X \to X | n \geq 1\}$ is a countable collection of hypercyclic operators, then their set of *common hypercyclic vectors*

$$\bigcap_{n=1}^{\infty} \mathcal{HC}(T_n)$$

is a dense G set.

- Salas (1999): If B is the unilateral backward shift, is the set of common hypercyclic vectors

$$\bigcap_{t>1} \mathcal{HC}(tB) \quad \neq \quad \phi?$$

Existence of a G_δ Set of Common Hypercyclic Vectors

- Abakumov & Gordon (2003): $\bigcap\limits_{1<t} \mathcal{HC}(tB)$ = the set of common hypercyclic vectors for tB is a dense G set.

Existence of a G_δ Set of Common Hypercyclic Vectors

- Abakumov & Gordon (2003): $\bigcap_{1<t} \mathcal{HC}(tB)$ = the set of common hypercyclic vectors for tB is a dense G set.

- Costakis & Sambarino (2004): Reproved the above result with a simpler proof by introducing a su cient condition for common hypercyclicity, and showed other applications.

Existence of a G_δ Set of Common Hypercyclic Vectors

- Abakumov & Gordon (2003): $\bigcap_{1<t} \mathcal{HC}(tB)$ = the set of common hypercyclic vectors for tB is a dense G set.

- Costakis & Sambarino (2004): Reproved the above result with a simpln(fo)28m23(T41)oic

Unilateral Weighted Backward Shifts on ℓ^p

$T : \ell^p \to \ell^p$ is said to be a *unilateral weighted backward shift* if there is a bounded positive weight sequence $\{w_j : j \geq 1\}$ such that

$$T(a_0, a_1, a_2, \ldots) = (w_1 a_1, w_2 a_2, w_3 a_3, \ldots).$$

$T : \ell^p \to \ell^p$ is said to be a *unilateral weighted backward shift* if there is a bounded positive weight sequence $\{w_j : j \geq 1\}$ such that

$$T(a_0, a_1, a_2, \ldots) = (w_1 a_1, w_2 a_2, w_3 a_3, \ldots).$$

- Salas (1995): A unilateral weighted backward shift T is hypercyclic if and only if $\sup\{w_1 w_2 \cdots w_n : n \geq 1\} = \infty$.

Unilateral Weighted Backward Shifts on ℓ^p

$T : \ell^p \to \ell^p$ is said to be a *unilateral weighted backward shift* if there is a bounded positive weight sequence $\{w_j : j \geq 1\}$ such that

$$T(a_0, a_1, a_2, \ldots) \;=\; (w_1 a_1, w_2 a_2, w_3 a_3, \ldots).$$

- Salas (1995): A unilateral weighted backward shift T is hypercyclic if and only if $\sup\{w_1 w_2 \cdots w_n : n \geq 1\} = \infty$.
- Grosse-Erdmann (2000): Generalizations to Frechet spaces.

$T : \ell^p \to \ell^p$ is a *bilateral weighted backward shift* if there is a bounded positive weight sequence $\{w_j : j \in \mathbb{Z}\}$ such that

$$T(\ldots, a_{-1}, \overbrace{a_0}^{\text{zeroth}}, a_1, \ldots) = (\ldots, w_{-1}a_{-1}, , w_0 a_0, \overbrace{w_1 a_1}^{\text{zeroth}}, w_2 a_2, \ldots).$$

Bilateral Weighted Shifts on ℓ^p

$T : \ell^p \to \ell^p$ is a <u>*bilateral weighted backward shift*</u> if there is a bounded positive weight sequence $\{w_j : j \in \mathbb{Z}\}$ such that

$$T(\ldots, a_{-1}, \overbrace{a_0}^{\text{zeroth}}, a_1, \ldots) = (\ldots, w_{-1}a_{-1}, , w_0 a_0, \overbrace{w_1 a_1}^{\text{zeroth}}, w_2 a_2, \ldots).$$

- Salas (1995): A bilateral weighted shift T is hypercyclic if and only if for any $\epsilon > 0$, and $q \in \mathbb{N}$, there is an arbitrarily large n such that whenever $|k| \leq q$,

$$\prod_{j=1}^{n} w_{k+j} > \frac{1}{\epsilon} \quad \text{and} \quad \prod_{j=0}^{n-1} w_{k-j} < \epsilon.$$

Paths of Hypercyclic Weighted Shifts on ℓ^p

- with Sanders (2009): Between any two hypercyclic unilateral weighted backward shifts, there is a path of such operators with a dense G_δ set of common hypercyclic vectors. Also, there is a path of such operators with no common hypercyclic vectors.

- with Sanders (2009): Between any two hypercyclic unilateral weighted backward shifts, there is a path of such operators with a dense G set of common hypercyclic vectors. Also, there is a path of such operators with no common hypercyclic vectors.

- Corollary: The hypercyclic unilateral weighted backward shifts form a path-connected subset in the operator algebra.

- with Sanders (2009): Between any two hypercyclic unilateral weighted backward shifts, there is a path of such operators with a dense G set of common hypercyclic vectors. Also, there is a path of such operators with no common hypercyclic vectors.
- Corollary: The hypercyclic unilateral weighted backward shifts form a path-connected subset in the operator algebra.
- The same holds true for bilateral weighted shifts.

Paths of Hypercyclic Weighted Shifts on ℓ^p

- with Sanders (2009): Between any two hypercyclic unilateral weighted backward shifts, there is a path of such operators with a dense G set of common hypercyclic vectors. Also, there is a path of such operators with no common hypercyclic vectors.

- Corollary: The hypercyclic unilateral weighted backward shifts form a path-connected subset in the operator algebra.

- The same holds true for bilateral weighted shifts.

Natural Question: Can we have \a lot" of operators in a path and yet their common hypercyclic vectors still form a dense G subset? What do we mean by \a lot"?

Existence of Hypercyclic Operators

- Ansari (1997) : For every separable, in nite dimensional Banach space X, there is a hypercyclic operator T in $B(X)$.

- Ansari (1997) : For every separable, infinite dimensional Banach space X, there is a hypercyclic operator T in $B(X)$.

Definition. A vector $x \in X$ is said to be a <u>*periodic point*</u> of an operator T in $B(X)$ if there is a positive integer n such that $T^n x = x$.

Definition. An operator on X is said to be <u>*chaotic*</u> if and only if it is hypercyclic and has a dense set of periodic points.

Existence of Hypercyclic Operators

- Ansari (1997) : For every separable, infinite dimensional Banach space X, there is a hypercyclic operator T in $B(X)$.

Definition. A vector $x \in X$ is said to be a *periodic point* of an operator T in $B(X)$ if there is a positive integer n such that $T^n x = x$.

Definition. An operator on X is said to be *chaotic* if and only if it is hypercyclic and has a dense set of periodic points.

- Bonet & Martínez-Gimenez & Peris (2001): There is a separable, infinite dimensional Banach space which admits no chaotic operator.

A Zero-One Law for Chaotic Operators

SOT = Strong Operator Topology of the operator algebra $B(X)$.

- (2002): For a separable, infinite dimensional Hilbert space H, the hypercyclic operators on H are SOT-dense in $B(H)$.

A Zero-One Law for Chaotic Operators

SOT = Strong Operator Topology of the operator algebra $B(X)$.

- (2002): For a separable, in nite dimensional Hilbert space H, the hypercyclic operators on H are SOT-dense in $B(H)$.
- with Bes (2003): The set of chaotic operators on a separable, in nite dimensional Banach space X is either empty or SOT-dense in $B(X)$.

A Zero-One Law for Chaotic Operators

SOT = Strong Operator Topology of the operator algebra $B(X)$.

- (2002): For a separable, in nite dimensional Hilbert space H, the hypercyclic operators on H are SOT-dense in $B(H)$.
- with Bes (2003): The set of chaotic operators on a separable, in nite dimensional Banach space X is either empty or SOT-dense in $B(X)$.

  Indeed, if $T \in B(X)$ is hypercyclic, then its conjugate class, or similarity orbit, $\{A^{-1}TA : A \text{ invertible on } X\}$ is SOT-dense in $B(X)$.

A Double Density Theorem

Let H be separable, infinite dimensional Hilbert space over $\mathbb{C}$.

- with Sanders (2011): There is a path of chaotic operators in $B(H)$ that is SOT-dense in $B(H)$, and each operator on the path shares the exact same set $\mathcal{G}$ of common hypercyclic vectors.

Let H be separable, infinite dimensional Hilbert space over $\mathbb{C}$.

- with Sanders (2011): There is a path of chaotic operators in $B(H)$ that is SOT-dense in $B(H)$, and each operator on the path shares the exact same set $\mathcal{G}$ of common hypercyclic vectors.
- Corollary: The path can be taken so that each operator along the path satisfies the hypercyclicity criterion.

A Double Density Theorem

Let H be separable, infinite dimensional Hilbert space over $\mathbb{C}$.

- with Sanders (2011): There is a path of chaotic operators in $B(H)$ that is SOT-dense in $B(H)$, and each operator on the path shares the exact same set $\mathcal{G}$ of common hypercyclic vectors.
- Corollary: The path can be taken so that each operator along the path satisfies the hypercyclicity criterion.
- Corollary: The hypercyclic operators in $B(H)$ are SOT-connected.
- Corollary: The hypercyclic operators T in $B(H)$ with $\mathcal{G} \subset \mathcal{HC}(T)$ are SOT-connected.

Similarity Orbits

For an operator $T : X \to X$ on a Banach space X, we let $\mathcal{S}(T) = \{A^{-1}TA \,|\, A \text{ invertible}\}$ be the similarity orbit of T.

Similarity Orbits

For an operator $T : X \to X$ on a Banach space X, we let $\mathcal{S}(T) = \{A^{-1}TA \,|\, A \text{ invertible}\}$ be the similarity orbit of T.

- with Sanders (2011): $\mathcal{S}(T)$ contains a path of operators which is SOT-dense in $B(X)$ and for which the set of common hypercyclic vectors for the whole path is a dense G set.

Similarity Orbits

For an operator $T : X \to X$ on a Banach space X, we let $\mathcal{S}(T) = \{A^{-1}TA \,|\, A \text{ invertible}\}$ be the similarity orbit of T.

- with Sanders (2011): $\mathcal{S}(T)$ contains a path of operators which is SOT-dense in $B(X)$ and for which the set of common hypercyclic vectors for the whole path is a dense G set.

Observations of some zero-one phenomenon:

Similarity Orbits

For an operator $T : X \to X$ on a Banach space X, we let $\mathcal{S}(T) = \{A^{-1}TA \,|\, A \text{ invertible}\}$ be the similarity orbit of T.

- with Sanders (2011): $\mathcal{S}(T)$ contains a path of operators which is SOT-dense in $B(X)$ and for which the set of common hypercyclic vectors for the whole path is a dense G set.

Observations of some zero-one phenomenon:

(1) If $\mathcal{HC}(T) = X \setminus \{0\}$, the set of common hypercyclic vectors for $\mathcal{S}(T)$ is also $X \setminus \{0\}$.

(2) If $\mathcal{HC}(T) \neq X \setminus \{0\}$, the set of common hypercyclic vectors for $\mathcal{S}(T)$ is empty.

For an operator T on a Hilbert space H, we let
$\mathcal{U}(T) =$

Unitary Orbits

For an operator T on a Hilbert space H, we let $\mathcal{U}(T) = \{U^{-1}TU : U \text{ unitary}\}$, the unitary orbit of T.

$\{\text{unitary operators on } H\}$ path connected $\implies \mathcal{U}(T)$ path connected.

Unitary Orbits

For an operator T on a Hilbert space H, we let $\mathcal{U}(T) = \{U^{-1}TU : U \text{ unitary}\}$, the unitary orbit of T.

$\{$unitary operators on $H\}$ path connected $\implies$ $\mathcal{U}(T)$ path connected.

Every operator in $\mathcal{U}(T)$ has the same norm as T. So $\mathcal{U}(T)$ does not contain a path that is SOT-dense in $B(H)$.

Unitary Orbits

For an operator T on a Hilbert space H, we let $\mathcal{U}(T) = \{U^{-1}TU : U \text{ unitary}\}$, the unitary orbit of T.

$\{$unitary operators on $H\}$ path connected $\implies$ $\mathcal{U}(T)$ path connected.

Every operator in $\mathcal{U}(T)$ has the same norm as T. So $\mathcal{U}(T)$ does not contain a path that is SOT-dense in $B(H)$.

- with Sanders (2012): If $T \in B(H)$ be hypercyclic, then $\mathcal{U}(T)$ contains a path $\mathcal{P}$ of operators so that $\overline{\mathcal{P}}^{SOT}$ contains $\mathcal{U}(T)$ and the common hypercyclic vectors for $\mathcal{P}$ is a dense G set.

A Zero-One Law for Orbital Limit Points

- Bourdon & Feldman (2003): If an orbit $\text{orb}(T, x)$ is somewhere dense in a Banach space X then the orbit $\text{orb}(T, x)$ is everywhere dense.

A Zero-One Law for Orbital Limit Points

- Bourdon & Feldman (2003): If an orbit orb(T, x) is somewhere dense in a Banach space X then the orbit orb(T, x) is everywhere dense.

- with Seceleanu (2012): Let $T : \ell^p \to \ell^p$ be a unilateral weighted backward shift. The following are equivalent:

(A) T is hypercyclic.

A Zero-One Law for Orbital Limit Points

- Bourdon & Feldman (2003): If an orbit orb(T, x) is somewhere dense in a Banach space X then the orbit orb(T, x) is everywhere dense.

- with Seceleanu (2012): Let $T : \ell^p \to \ell^p$ be a unilateral weighted backward shift. The following are equivalent:

(A) T is hypercyclic.

(B) There is a vector whose orbit has a nonzero limit point.

A Zero-One Law for Orbital Limit Points

- Bourdon & Feldman (2003): If an orbit $\mathrm{orb}(T, x)$ is somewhere dense in a Banach space X then the orbit $\mathrm{orb}(T, x)$ is everywhere dense.

- with Seceleanu (2012): Let $T : \ell^p \to \ell^p$ be a unilateral weighted backward shift. The following are equivalent:

(A) T is hypercyclic.
(B) There is a vector whose orbit has a nonzero limit point.
(C) There is a vector whose orbit has a nonzero weak limit point.

A Zero-One Law for Orbital Limit Points

- Bourdon & Feldman (2003): If an orbit orb(T, x) is somewhere dense in a Banach space X then the orbit orb(T, x) is everywhere dense.

- with Seceleanu (2012): Let $T : \ell^p \to \ell^p$ be a unilateral weighted backward shift. The following are equivalent:

(A) T is hypercyclic.
(B) There is a vector whose orbit has a nonzero limit point.
(C) There is a vector whose orbit has a nonzero weak limit point.
(D) There is a vector whose orbit has infinitely many members contained in an open ball whose closure avoids the origin.

A Zero-One Law for Orbital Limit Points

- Bourdon & Feldman (2003): If an orbit orb(T, x) is somewhere dense in a Banach space X then the orbit orb(T, x) is everywhere dense.

- with Seceleanu (2012): Let $T : \ell^p \to \ell^p$ be a unilateral weighted backward shift. The following are equivalent:

(A) T is hypercyclic.
(B) There is a vector whose orbit has a nonzero limit point.
(C) There is a vector whose orbit has a nonzero weak limit point.
(D)

A Zero-One Law for Orbital Limit Points

- Bourdon & Feldman (2003): If an orbit $\mathrm{orb}(T, x)$ is somewhere dense in a Banach space X then the orbit $\mathrm{orb}(T, x)$ is everywhere dense.

- with Seceleanu (2012): Let $T : \ell^p \to \ell^p$ be a unilateral weighted backward shift. The following are equivalent:

(A) T is hypercyclic.
(B) There is a vector whose orbit has a nonzero limit point.
(C) There is a vector whose orbit has a nonzero weak limit point.
(D) There is a vector whose orbit has infinitely many members contained in an open ball whose closure avoids the origin.

Corollary: T is not hypercyclic iff every $\mathrm{orb}(T, x) \cup \{0\}$ is closed.

Remark: (A), (B), (D) are equivalent for bilateral weighted shifts.

A Zero-One Law for Orbital Limit Points

- Bourdon & Feldman (2003): If an orbit orb(T, x) is somewhere dense in a Banach space X then the orbit orb(T, x) is everywhere dense.

- with Seceleanu (2012): Let $T : \ell^p \to \ell^p$ be a unilateral weighted backward shift. The following are equivalent:

(A) T is hypercyclic.
(B) There is a vector whose orbit has a nonzero limit point.
(C) There is a vector whose orbit has a nonzero weak limit point.
(D) There is a vector whose orbit has in nitely many members contained in an open ball whose closure avoids the origin.

Corollary: T is not hypercyclic i every orb$(T, x) \cup \{0\}$ is closed.

Remark: (A), (B), (D) are equivalent for bilateral weighted shifts.

- with Sanders (2004): A unilateral weighted backward shift is hypercyclic if and only if it is weakly hypercyclic. But, there is a bilateral weighted shift that is weakly hypercyclic but not hypercyclic.

A Remark on Theorem

If $\mathrm{orb}(T, x)$ has a nonzero limit point, we can only conclude T is hypercyclic but we cannot conclude that x is a hypercyclic vector, and in fact not even a cyclic vector.

A vector x is a *cyclic vector* for T, if $\mathrm{span}\,\mathrm{orb}(T, x)$ is dense in X.

A Remark on Theorem

If orb(T, x) has a nonzero limit point, we can only conclude T is hypercyclic but we cannot conclude that x is a hypercyclic vector, and in fact not even a cyclic vector.

A vector x is a *cyclic vector* for T, if span orb(T, x) is dense in X.

Let (e_n) be the canonical basis of ℓ^p.

- with Seceleanu (preprint, 2013): The vector x is a cyclic vector for T, if

  (1) the weight $(w_j)_{j=1}^{\infty}$ of T is bounded below, and

  (2) orb(T, x) has a nonzero limit point f given by $f = a_0 e_o + \cdots + a_n e_n$ (nite sum) for some scalars a_j.

A Remark on Theorem

If $\text{orb}(T, x)$ has a nonzero limit point, we can only conclude T is hypercyclic but we cannot conclude that x is a hypercyclic vector, and in fact not even a cyclic vector.

A vector x is a *cyclic vector* for T, if span $\text{orb}(T, x)$ is dense in X.

Let (e_n) be the canonical basis of ℓ^p.

- with Seceleanu (preprint, 2013): The vector x is a cyclic vector for T, if

  (1) the weight $(w_j)_{j=1}^{\infty}$ of T is bounded below, and

  (2) $\text{orb}(T, x)$ has a nonzero limit point f given by $f = a_0 e_o + \cdots + a_n e_n$ (nite sum) for some scalars a_j.

There are examples to show both (1) and (2) are needed for x to be a cyclic vector.

Proof of \(B) $\Longrightarrow$ (A)"

Suppose there exist a vector $x = (x_0, x_1, x_2, \ldots) \in \ell^p$ and a non-zero vector $f = (f_0, f_1, f_2, \ldots) \in \ell^p$ such that f is a limit point of the orbit $\mathrm{Orb}(T, x)$.

Suppose there exist a vector $x = (x_0, x_1, x_2, \ldots) \in \ell^p$ and a non-zero vector $f = (f_0, f_1, f_2, \ldots) \in \ell^p$ such that f is a limit point of the orbit $\mathrm{Orb}(T, x)$.

Since $f_j \neq 0$ for some $j \geq 0$, we assume without loss of generality that $f_0 \neq 0$. Hence there exist an increasing sequence $\{n_k : k \geq 1\} \subset \mathbb{N}$ and an integer $N > 0$ such that

$$\| T^{n_k} x - f \| < \frac{1}{2^k} < \frac{|f_0|}{2},$$

for all $k \geq N$. Then

$$T^{n_k} x = T^{n_k}(x_0, x_1, x_2, \ldots) = (w_1 \cdots w_{n_k} x_{n_k}, \ldots).$$

Suppose there exist a vector $x = (x_0, x_1, x_2, \ldots) \in \ell^p$ and a non-zero vector $f = (f_0, f_1, f_2, \ldots) \in \ell^p$ such that f is a limit point of the orbit $\mathrm{Orb}(T, x)$.
Since $f_j \neq 0$ for some $j \geq 0$, we assume without loss of generality that $f_0 \neq 0$. Hence there exist an increasing sequence $\{n_k : k \geq 1\} \subset \mathbb{N}$ and an integer $N > 0$ such that

$$\|T^{n_k}x - f\| < \frac{1}{2^k} < \frac{|f_0|}{2},$$

for all $k \geq N$. Then

$$T^{n_k}x = T^{n_k}(x_0, x_1, x_2, \ldots) = (w_1 \cdots w_{n_k} x_{n_k}, \ldots).$$

Hence $\|T^{n_k}x - f\| \geq |w_1 \cdots w_{n_k} x_{n_k} - f_0|$. So there exists a sequence $\{n_k : k \geq 1\}$ such that

$$|w_1 \cdots w_{n_k} x_{n_k} - f_0| < |f_0|/2,$$

for all $k \geq N$.

\(B) $\Longrightarrow$ (A)" Completed

Thus $|f_0|/2 < |w_1 \cdots w_{n_k} x_{n_k}|$ and so $\frac{|f_0|}{2(w_1 \cdots w_{n_k})} < |x_{n_k}|$ for all $k \geq N$. Hence we get that

$$\frac{|f_0|^p}{2^p (w_1 \cdots w_{n_k})^p} < |x_{n_k}|^p, \text{ for all } k \geq N \ .$$

Now since $x \in \ell^p$ we have

$$\frac{|f_0|^p}{2^p} \sum_{k=N}^{\infty} \frac{1}{(w_1 \cdots w_{n_k})^p} \leq \sum_{k=N}^{\infty} |x_{n_k}|^p \leq \|x\|^p < \infty.$$

\(B) $\Longrightarrow$ (A)" Completed

Thus $|f_0|/2 < |w_1 \cdots w_{n_k} x_{n_k}|$ and so $\frac{|f_0|}{2(w_1 \cdots w_{n_k})} < |x_{n_k}|$ for all $k \geq N$. Hence we get that

$$\frac{|f_0|^p}{2^p (w_1 \cdots w_{n_k})^p} < |x_{n_k}|^p, \text{ for all } k \geq N \ .$$

Now since $x \in \ell^p$ we have

$$\frac{|f_0|^p}{2^p} \sum_{k=N}^{\infty} \frac{1}{(w_1 \cdots w_{n_k})^p} \leq \sum_{k=N}^{\infty} |x_{n_k}|^p \leq \|x\|^p < \infty.$$

It follows that $\frac{1}{(w_1 ::: w_{n_k})^p} \to 0$. That is, there exists an increasing sequence $\{n_k\}$ such that $w_1 \cdots w_{n_k} \to \infty$ as $k \to \infty$.

Thus $|f_0|/2 < |w_1 \cdots w_{n_k} x_{n_k}|$ and so $\frac{|f_0|}{2(w_1 \cdots w_{n_k})} < |x_{n_k}|$ for all $k \geq N$. Hence we get that

$$\frac{|f_0|^p}{2^p (w_1 \cdots w_{n_k})^p} < |x_{n_k}|^p, \text{ for all } k \geq N \ .$$

Now since $x \in \ell^p$ we have

$$\frac{|f_0|^p}{2^p} \sum_{k=N}^{\infty} \qquad 1$$

Recall: A Zero-One Law for Orbital Limit Points

- with Seceleanu (2012): Let $T : \ell^p \to \ell^p$ be a unilateral weighted backward shift. The following are equivalent:

(A) T is hypercyclic.

(B) There is a vector whose orbit has a nonzero limit point.

(C) There is a vector whose orbit has a nonzero weak limit point.

(D) There is a vector whose orbit has in nitely many members contained in an open ball whose closure avoids the origin.

Proof of \(C) $\implies$ (B)"

Let $x = (x_0, x_1, x_2, \ldots) \in \ell^p$ be a vector whose $\mathrm{Orb}(T, x)$ has $f = (f_0, f_1, f_2, \ldots) \in \ell^p$ as a non-zero <u>weak</u> limit point, with f_k)

Proof of \(C) $\Longrightarrow$ (B)"

Let $x = (x_0, x_1, x_2, \ldots) \in \ell^p$ be a vector whose $\mathrm{Orb}(T, x)$ has $f = (f_0, f_1, f_2, \ldots) \in \ell^p$ as a non-zero <u>weak</u> limit point, with $f_k \neq 0$.

Considering the weakly open sets that contain f, we get that for all $j \geq 1$ there exists an $n_j \geq 1$ such that $|\langle T^{n_j} x - f, e_k \rangle| < \frac{1}{j}$.

That is $\left| w_{k+1} \cdots w_{k+n_j} x_{k+n_j} - f_k \right| < \frac{1}{j}$, for all $j \geq 1$.

Next, we inductively pick a subsequence $\{n_{j_k}\}$ of $\{n_j\}$ as follows:
1. Let $j_1 = 1$.
2. Once we have chosen j_m we pick $j_{m+1} > j_m$ such that

$$k + n_{j_m} < n_{j_{m+1}} \text{ and } \sum_{i=j_{m+1}}^{\infty} |x_{k+n_i}|^p \leq \frac{1}{j_m \cdot \|T\|^{p \cdot n_{j_m}}}.$$

Thus we can assume, by taking a subsequence if necessary, that

$$\{n_j\} \text{ satis es } k + n_j < n_{j+1} \text{ and } \sum_{i=j+1}^{\infty} |x_{k+n_i}|^p \leq \frac{1}{j \cdot \|T\|^{p \cdot n_j}}.$$

\(C) $\implies$ (B)" Continued

Let $y = \sum_{i=1}^{\infty} x_{k+n_i} \cdot e_{k+n_i}$. Clearly y is in ℓ^p, because x is.

Then $T^{n_m} y = \sum_{i=1}^{\infty} x_{k+n_i} \cdot T^{n_m} e_{k+n_i}$. But $k + n_i < n_{i+1}$ for all $i \geq 1$ and so $k + n_i < n_m$ for all $i < m$. Thus since T is a unilateral backward shift we conclude that $T^{n_m} y = \sum_{i=m}^{\infty} x_{k+n_i} \cdot T^{n_m} e_{k+n_i}$.

\(C) $\Longrightarrow$ (B)" Continued

Let $y = \sum_{i=1}^{\infty} x_{k+n_i} \cdot e_{k+n_i}$. Clearly y is in ℓ^p, because x is.

Then $T^{n_m} y = \sum_{i=1}^{\infty} x_{k+n_i} \cdot T^{n_m} e_{k+n_i}$. But $k + n_i < n_{i+1}$ for all $i \geq 1$ and so $k + n_i < n_m$ for all $i < m$. Thus since T is a unilateral backward shift we conclude that $T^{n_m} y = \sum_{i=m}^{\infty} x_{k+n_i} \cdot T^{n_m} e_{k+n_i}$.

Furthermore, since the vectors $T^{n_m} e_{k+n_i}$ and $T^{n_m} e_{k+n_j}$ have disjoint support for $i \neq j$, that is $\widehat{T^{n_m} e_{k+n_i}}(s) = 0$ whenever $\widehat{T^{n_m} e_{k+n_j}}(s) \neq 0$, we have that

$$
\begin{aligned}
& \|T^{n_m} y - f_k e_k\| \\
\leq \quad & \|(w_{k+1} \cdots w_{k+n_m} x_{k+n_m} - f_k) \cdot e_k\| + \left\| \sum_{i=m+1}^{\infty} x_{k+n_i} \cdot T^{n_m} e_{k+n_i} \right\|
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \| T^{n_m} y - f_k e_k \| \\
\leq \; & |w_{k+1} \cdots w_{k+n_m} x_{k+n_m} - f_k| + \left[\sum_{i=m+1}^{\infty} |x_{k+n_i}|^p \cdot \| T^{n_m} e_{k+n_i} \|^p \right]^{1=p} \\
\leq \; & \frac{1}{m} + \left[\sum_{i=m+1}^{\infty} |x_{k+n_i}|^p \cdot \| T \|^{p \cdot n_m} \right]^{1=p} \leq \frac{1}{m} + \frac{1}{\sqrt[p]{m}} \to 0 \;\; as \; m \to \infty.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|T^{n_m}y - f_k e_k\| \\
\leq\ & |w_{k+1}\cdots w_{k+n_m}x_{k+n_m} - f_k| + \left[\sum_{i=m+1}^{\infty} |x_{k+n_i}|^p \cdot \|T^{n_m}e_{k+n_i}\|^p\right]^{1=p} \\
\leq\ & \frac{1}{m} + \left[\sum_{i=m+1}^{\infty} |x_{k+n_i}|^p \cdot \|T\|^{p\cdot n_m}\right]^{1=p} \leq \frac{1}{m} + \frac{1}{\sqrt[p]{m}} \to 0 \ \text{ as } m \to \infty.
\end{aligned}
$$

Thus $T^{n_m}y \to f_k e_k$ in norm as $m \to \infty$, where $f_k e_k \neq 0$ in ℓ^p, and hence $\mathrm{Orb}(T, y)$ has a non-zero limit point. $\square$

Bergman Spaces

Let   be a region in $\mathbb{C}$ and $H^\infty(\ )$ be the algebra of all bounded analytic functions on .

Let $A^2(\ ) = \{f :\ \to \mathbb{C} \,|\, f$ analytic, and $\int |f|^2\,dA < \infty\}$ be the Bergman space.

If $\varphi \in H^\infty(\ )$, then we de ne $M_\varphi : A^2(\ ) \to A^2(\ )$ by $M_\varphi f = \varphi f$.

Let be a region in $\mathbb{C}$ and $H^{\infty}(\ )$ be the algebra of all bounded analytic functions on .

Let $A^2(\ ) = \{f : \ \to \mathbb{C} \,|\, f$ analytic, and $\int \ |f|^2 \, dA < \infty\}$ be the Bergman space.

If $\varphi \in H^{\infty}(\ )$, then we de ne $M_{\cdot} : A^2(\ ) \to A^2(\ )$ by $M_{\cdot} f = \varphi f$.

- Godefroy & Shapiro (1991): The adjoint operator $M_{\cdot}^{*} : A^2(\ ) \to A^2(\ )$ is hypercyclic if and only if $\varphi(\ )$ intersects the unit circle.

A Zero-One Law for Adjoint Multiplication Operators

Let $\varphi \in H^\infty(\quad)$ be a nonconstant function, and
$M_{\varphi} : A^2(\quad) \to A^2(\quad)$.

- with Seceleanu (2012): The following are equivalent.

(A) M_{φ}^* is hypercyclic.

Let $\varphi \in H^\infty(\ \ )$ be a nonconstant function, and $M_{\varphi} : A^2(\ \ ) \to A^2(\ \ )$.

- with Seceleanu (2012): The following are equivalent.

(A) M_{φ}^* is hypercyclic.

(B) M_{φ}^* has an orbit with a nonzero limit point.

A Zero-One Law for Adjoint Multiplication Operators

Let $\varphi \in H^{\infty}(\ \ )$ be a nonconstant function, and
$M_{\varphi} : A^2(\ \ ) \to A^2(\ \ )$.

- with Seceleanu (2012): The following are equivalent.

(A) M_{φ}^{*} is hypercyclic.

(B) M_{φ}^{*} has an orbit with a nonzero limit point.

(C) M_{φ}^{*} has an orbit which has in nitely many members contained in an open ball whose closure avoids the origin.

What about the Hardy Space?

Let $\mathbb{D}$ be the open unit disk, and let

$$H^2 = \left\{ f : \mathbb{D} \to \mathbb{D} \,|\, f(z) = \sum_0^\infty a_n z^n \text{ analytic and } \sum_0^\infty |a_n|^2 < \infty \right\}$$

be the Hardy space.

- with Seceleanu (2012): The result for the Bergman space holds true for the Hardy space.

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map.

De ne $C_{\prime} : H^2 \to \quad \cup \quad \Rightarrow \quad \infty \quad /$

What about the Hardy Space?

Let $\mathbb{D}$ be the open unit disk, and let

$$H^2 = \left\{ f : \mathbb{D} \to \mathbb{D} \,|\, f(z) = \sum_0^\infty a_n z^n \text{ analytic and } \sum_0^\infty |a_n|^2 < \infty \right\}$$

be the Hardy space.

- with Seceleanu (2012): The result for the Bergman space holds true for the Hardy space.

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map.

Define $C_\varphi : H^2 \to H^2$ by $C_\varphi f = f \circ \varphi$.

- with Seceleanu (2012): If $\alpha > 0$ is an irrational number, and $\varphi(z) = e^{2\pi i\alpha} z$, then C_φ has an orbit with the identity function $\psi(z) \equiv z$ as a nonzero limit point, but C_φ is not hypercyclic.